A proposed theoretical basis for a MOND cosmology: I The inertial frame?

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Abstract

This paper is essentially a speculation on the realization of Mach's Principle, and we came to the details of the present analysis via the formulation of two questions: *(a) Can a globally inertial space & time be associated with a non-trivial global matter distribution? (b) If so, what are the general properties of such a global distribution?*

These questions are addressed within the context of an extremely simple model universe consisting of particles possessing only the property of enumerability existing in a formless continuum. Since there are no pre-specified ideas of clocks and rods in this model universe, we are forced into two fundamental considerations, these being: *What invariant meanings can be given to the concepts of spatial displacement and elapsed time in this model universe?*
Briefly, these questions are answered as follows: the spatial displacement of a particle is defined in terms of its changed relationship with the particle ensemble as a whole - this is similar to the man walking down a street who can estimate the length of his walk by reference to his changed view of the street. Once the concept of invariant spatial displacement is established, a corresponding concept of elapsed time then emerges in a natural way as 'process' within the system.

Thus, unlike for example, general relativity, which can be considered as a theory describing the behaviour of specified clocks and rods in the presence of matter, the present analysis can be considered as a rudimentary - but fundamental - theory of what underlies the concepts of clocks and rods in a material universe. In answer to the original two questions, this theory tells us that a globally inertial space & time *can* be associated with a non-trivial global matter distribution, and that this distribution is *necessarily* fractal with $D = 2$.

This latter result is compared with the results of modern surveys of galaxy distributions which find that such distributions are quasi-fractal with $D \approx 2$ on

the small-to-medium scales, with the situation on the medium-to-large scales being a topic of considerable debate. Accordingly, and bearing in mind the extreme simplicity of the model considered, the observational evidence is consistent with the interpretation that the analysed point-of-view captures the cosmic reality to a good first-order approximation. We consider the implications of these results.

Inertia – **Mach** – **MOND** – **Fractal** – **Cosmology**

1   Introduction

The ideas underlying what is now known as 'Mach's Principle' can be traced to Berkeley (1710, 1721) for which a good contemporary discussion can be found in Popper (1953). Berkeley's essential insight, formulated as a rejection of Newton's ideas of absolute space, was that the motion of any object had no meaning except insofar as that motion was referred to some other object, or set of objects. Mach (1960, reprint of 1883 German edition) went much further than Berkeley when he said *I have remained to the present day the only one who insists upon referring the law of inertia to the earth and, in the case of motions of great spatial and temporal extent, to the fixed stars.* In this way, Mach formulated the idea that, ultimately, inertial frames should be defined with respect to the average rest frame of the visible universe.

It is a matter of history that Einstein was greatly influenced by Mach's ideas as expressed in the latter's *The Science of Mechanics ...* (see for example Pais 1982) and believed that they were incorporated in his field equations so long as space was closed (Einstein 1950). The modern general relativistic analysis gives detailed quantitative support to this latter view, showing how Mach's Principle can be considered to arise as a consequence of the field equations when appropriate conditions are specified on an initial hypersurface in a closed evolving universe. In fact, in answer to Mach's question asking what would happen to inertia if mass was progressively removed from the universe, Lynden-Bell, Katz & Bicak (1995) point out that, in a *closed* Friedmann universe the maximum radius of this closed universe and the duration of its existence both shrink to zero as mass is progressively removed. Thus, it is a matter of record that a satisfactory incorporation of Mach's Principle within general relativity can be attained when the constraint of closure is imposed.

However, there is a hardline point of view: in practice, when we talk of physical space (and the space composed of the set of all inertial frames in particular), we mean a space in which *distances* and *displacements* can be determined - but these concepts only have any meaning insofar as they refer to relationships

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within material systems. Likewise, when we refer to elapsed physical time, we mean a measurable degree of ordered change (process) occurring within a given physical system. Thus, all our concepts of measurable 'space & time' are irreducibly connected to the existence of material systems and to process within such systems - which is why the closed Friedmann solutions are so attractive. However, from this, we can also choose to conclude that any theory (for example, general relativity notwithstanding its closed Friedmann solutions) that allows an internally consistent discussion of an empty inertial spacetime must be non-fundamental at even the classical level.

To progress, we take the point of view that, since all our concepts of measurable 'space & time' are irreducibly connected to the existence of material systems and to process within such systems, then these concepts are, in essence, *metaphors* for the relationships that exist between the individual particles (whatever these might be) within these material systems. Since the most simple conception of physical space & time is that provided by inertial space & time, we are then led to two simple questions:

Is it possible to associate a globally inertial space & time with a non-trivial global matter distribution and, if it is, what are the fundamental properties of this distribution?

In the context of the simple model analysed, the present paper finds definitive answers to these questions so that:

- *A globally inertial space & time can be associated with a non-trivial global distribution of matter;*

- *This global distribution is necessarily fractal with $D = 2$.*

In the following, we construct a simple model universe, analyse it within the context of the basic questions posed, and consider other significant matters which arise naturally within the course of the development.

2   General overview

We start from the position that conceptions of an *empty* inertial spatio-temporal continuum are essentially non-physical, and are incapable of providing sound foundations for fundamental theory. The fact that we have apparently successful theories based exactly on such conceptions does not conflict with this statement - so long as we accept that, in such cases, the empty inertial spatio-temporal continuum is understood to be a metaphor for a deeper reality in which the metric (or inertial) properties of this spatio-temporal continuum are somehow projected out of an *unaccounted-for* universal distribution of material. For example, according to this view, the fact that general relativity admits an empty inertial spatio-temporal continuum as a special case (and was actually originally derived as a generalization of such a construct) implies that it is based upon such a metaphor - and is therefore, according to this view, not sufficiently primitive to act as a basis from which fundamental theories of cosmology can be constructed.

By starting with a model universe consisting of objects which have no other properties except identity (and hence enumerability) existing in a formless continuum, we show how it is possible to project spatio-temporal metric properties from the objects onto the continuum. By considering idealized dynamical equilibrium conditions (which arise as a limiting case of a particular free parameter going to zero), we are then able to show how a globally inertial spatio-temporal continuum is necessarily identified with a material distribution which has a fractal dimension $D = 2$ in this projected space. This is a striking result since it bears a very close resemblance to the cosmic reality for the low-to-medium redshift regime.

However, this idealized limiting case material distribution is distinguished from an ordinary material distribution in the sense that the individual particles of which it is comprised are each in a state of arbitrarily directed motion, but with equal-magnitude velocities for all particles - and in this sense is more like a quasi-photon gas distribution. For this reason, we interpret the distribution as

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a rudimentary representation of an inertial material vacuum, and present it as the appropriate physical background within which gravitational processes (as conventionally understood) can be described as point-source perturbations of an inertial spatio-temporal-material background. We briefly discuss how such processes can arise.

2.1 Overview of the non-relativistic formalism

In order to clarify the central arguments and to minimize conceptual problems in this initial development, we assume that the model universe is stationary in the sense that the overall statistical properties of the material distribution do not evolve in any way. Whilst this was intended merely as a simplifying assumption, it has the fundamental effect of making the development inherently non-relativistic (in sense that the system evolves within a curved metric three-space, rather than being a geodesic structure within a spacetime continuum).

The latter consequence arises in the following way: since the model universe is assumed to be stationary, then there is no requirement to import a pre-determined concept of 'time' into the discussion at the beginning - although the qualitative notion of a generalized 'temporal ordering' is assumed. The arguments used then lead to a formal model which allows the natural introduction of a generalized temporal ordering parameter, and this formal model is invariant with respect to any transformation of this latter parameter which leaves the absolute ordering of events unchanged. This arbitrariness implies that the formal model is incomplete, and can only be completed by the imposition of an additional condition which constrains the temporal ordering parameter to be identifiable with some model of physical time. It is then found that such a model of physical time, defined in terms of 'system process', arises automatically from the assumed isotropies within the system. In summary, the assumption of stationarity leads to the emergent concept of a physical 'spatio-temporal continuum' which partitions into a metric three-space together with a distinct model of physical time defined in terms of ordered material process in the metric three-

space. The fractal $D = 2$ inertial universe then arises as an idealized limiting case.

2.2   Overview of the relativistic formalism

The relativistic formalism arises as a natural consequence of relaxing the constraint of a *stationary* universe. The formalism is not considered in any detail here but, briefly, its development can be described as follows: if the universe is not stationary, then it is evolving - and this implies the need for a pre-determined concept of 'time' to be included in the discussion at the outset. If this is defined in any of the ways which are, in practice, familiar to us then we can reasonably refer to it as 'local process time'. Arguments which exactly parallel those used in the stationary universe case considered in detail here then lead to a situation which is identical to that encountered in the Lagrangian formulation of General Relativity: in that historical case, the equations of motion include a local coordinate time (which corresponds to our local process time) together with a global temporal ordering parameter, and the equations of motion are invariant with respect to any transformation of this latter parameter which leaves the ordering of 'spacetime' events unchanged. This implies that the equations of motion are incomplete - and the situation is resolved there by defining the global temporal ordering parameter to be 'particle proper time'. The solution we adopt for our evolving universe case is formally identical, so that everything is described in terms of a metric 'spacetime'. By considering idealized dynamical equilibrium conditions, we are led to the concept of an inertial 'spacetime' which is identical to the spacetime of special relativity - except that it is now irreducibly associated with a fractally distributed *relativistic* 'photon gas'.

3   The starting point

In §1, we offered the view that the fundamental significance of Mach's Principle arises from its implication of the impossibility of defining inertial frames in the

absence of material; or, as a generalization, that it is *impossible* to conceive of a physical spatio-temporal continuum in the absence of material. It follows from this that, if we are to arrive at a consistent and fundamental implementation of Mach's Principle, then we need a theory of the world according to which (roughly speaking) notions of the spatio-temporal continuum are somehow projected out of primary relationships between objects. In other words, we require a theory in which notions of metrical space & time are to be considered as metaphors for these primary relationships. Our starting point is to consider the calibration of a radial measure which conforms to these ideas.

Consider the following perfectly conventional procedure which assumes that we 'know' what is meant by a given radial displacement, R say. On a large enough scale ($> 10^8$ light years, say), we can reasonably assume it is possible to write down a relationship describing the amount of mass contained within a given spherical volume: say

$$M = U(R), \tag{1}$$

where U is, in principle, determinable. Of course, a classical description of this type ignores the discrete nature of real material; however, overlooking this point, such a description is completely conventional and unremarkable. Because M obviously increases as R increases, then U is said to be monotonic, with the consequence that the above relationship can be inverted to give

$$R = G(M) \tag{2}$$

which, because (1) is unremarkable, is also unremarkable.

In the conventional view, (1) is logically prior to (2); however, it is perfectly possible to reverse the logical priority of (1) and (2) so that, in effect, we can choose to *define* the radial measure in terms of (2) rather than assume that it is known by some independent means. If this is done then, immediately, we have made it impossible to conceive of radial measure in the absence of material.

With this as a starting point, we are able to construct a completely Machian Cosmology in a way outlined in the following sections.

4   A discrete model universe

The model universe is intended as an idealization of our actual universe, and is defined as follows:

- *it consists of an infinity of identical, but labelled, discrete material particles which are primitive, possessing no other properties beyond being countable;*

- *'time' is to be understood, in a qualitative way, as a measure of process or ordered change in the model universe;*

- *there is at least one origin about which the distribution of material particles is statistically isotropic - meaning that the results of sampling along arbitrary lines of sight over sufficiently long characteristic 'times' are independent of the directions of lines of sight;*

- *the distribution of material is statistically stationary - meaning that the results of sampling along arbitrary lines of sight over sufficiently long characteristic 'times' are independent of sampling epoch.*

Although concepts of invariant spatio-temporal measurement are implicitly assumed to exist in this model universe, we make no apriori assumptions about their quantitative definition, but require that such definitions should arise naturally from the structure of the model universe and from the following analysis.

4.1   The invariant calibration of a radial coordinate in terms of counting primitive objects.

At (2), we have already introduced, in a qualitative way, the idea that the radial magnitude of a given sphere can be *defined* in terms of the amount of material contained within that sphere and, in this section, we seek to make this idea

more rigorous. To this end, we note that the most primitive invariant that can be conceived is that based on the counting of objects in a countable set, and we show how this fundamental idea can be used to define the concept of invariant distance in the model universe.

The isotropy properties assumed for the model universe imply that it is statistically spherically symmetric about the chosen origin. If, for the sake of simplicity, it is assumed that the characteristic sampling times over which the assumed statistical isotropies become exact are infinitesimal, then the idea of statistical spherical symmetry gives way to the idea of exact spherical symmetry - thereby allowing the idea of some kind of rotationally invariant radial coordinate to exist. As a first step towards defining such an idea, suppose only that the means exists to define a succession of nested spheres, $S_1 \subset S_2 \subset ... \subset S_p$, about the chosen origin; since the model universe with infinitesimal characteristic sampling times is stationary, then the flux of particles across the spheres is such that these spheres will always contain fixed numbers of particles, say $N_1, N_2, ..., N_p$ respectively.

Since the only invariant quantity associated with any given sphere, S say, is the *number* of material particles contained within it, N say, then the only way to associate an invariant radial coordinate, r say, with S is to *define* it according to $r = r_0 f(N)$ where r_0 is a fixed scale-constant having units of 'length', and the function f is restricted by the requirements $f(N_a) > f(N_b)$ whenever $N_a > N_b$, $f(N) > 0$ for all $N > 0$, and $f(0) = 0$. To summarize, an invariant calibration of a radial coordinate in the model universe is given by $r = r_0 f(N)$ where:

- $f(N_a) > f(N_b)$ whenever $N_a > N_b$;

- $f(N) > 0$ for all $N > 0$ and $f(0) = 0$.

Once a radial coordinate has been invariantly calibrated, it is a matter of routine to define a rectangular coordinate system based upon this radial calibration; this is taken as done for the remainder of this paper.

4.2   The mass model

At this stage, since no notion of 'inertial frame' has been introduced then the idea of 'inertial mass' cannot be defined. However, we have assumed the model universe to be composed of a countable infinity of labelled - but otherwise indistinguishable - material particles so that we can associate with each individual particle a property called 'mass' which quantifies the amount of material in the particle, and is represented by a scale-constant, m_0 say, having units of 'mass'.

The radial parameter about any point is *defined* by $r = r_0 f(N)$; since this function is constrained to be monotonic, then its inverse exists so that, by definition, $N = f^{-1}(r/r_0)$; suppose we now introduce the scale-constant m_0, then $Nm_0 = m_0 f^{-1}(r/r_0) \equiv M(r)$ can be *interpreted* as quantifying the total amount of material inside a sphere of radius r centred on the assumed origin. Although $r = r_0 f(N)$ and $M(r) = Nm_0$ are equivalent, the development which follows is based upon using $M(r)$ as a description of the mass distribution given as a function of an invariant radial distance parameter, r, of undefined calibration.

It is clear from the foregoing discussion that r is defined as a necessarily discrete parameter. However, to enable the use of familiar techniques, it will hereafter be supposed that r represents a continuum - it being understood that a fully consistent treatment will require the use of discrete mathematics throughout.

5   The absolute magnitudes of arbitrary displacements in the model universe

We have so far defined, in general terms, an invariant radial coordinate calibration procedure in terms of the radial distribution of material valid from the assumed origin, and have noted that such a procedure allows a routine definition of orthogonal coordinate axes. Whilst this process has provided a means by which arbitrary displacements can be described relative to the global material

distribution, it does not provide the means by which an invariant *magnitude* can be assigned to such displacements - that is, there is no metric defined for the model universe. In the following, we show how the notion of 'metric' can be considered to be projected from the mass distribution.

5.1 Change in perspective as an general indicator of displacement in a material universe

In order to understand how the notion of 'metric' can be defined, we begin by noting the following empirical circumstances from our familiar world:

- In reality, an observer displaced from one point to another recognizes the fact of his own spatial displacement by reference to his changed perspective of his (usually local) material universe;

- the magnitude of this change in perspective provides a measure of the magnitude of his own spatial displacement.

To be more specific, consider an idealized scene consisting of a distributed set of many labelled points all in a static relationship with respect to each other, plus an observer of this scene. Since the labelled points are in a static relationship to each other, then a subset of them can be used to define a reference frame within which all of the other labelled points in the scene occupy fixed positions. The specification of the observer's directions-of-view onto any two of the labelled points in this scene (which are not colinear with him!) uniquely fixes the observer's position and hence his *perspective* of the whole scene. Correspondingly, the starting and finishing points of any journey undertaken by the observer can be specified by the initial and final directions-of-view onto each of the two chosen labelled points, and the journey itself can be given an invariant description purely in terms of these initial and final directions-of-view conditions - that is, in terms of the observer's changed perspective of the whole scene.

To summarize, an observer's perspective of a scene can be considered defined by his coordinate position in the defined reference frame plus a direction of view

onto a specified labelled point within the scene, and an invariant description of any journey made by the observer of the scene can given in terms of change in this perspective. In the following, we show exactly how the concept of 'change in perspective' can be used to associated invariant magnitudes to coordinate displacements in the model universe.

5.2 Perspective in the model universe

Since, in the present case, we are seeking to give invariant meaning to the displacement of an arbitrarily chosen particle in the model universe, then we replace the journeying observer of the foregoing static scene by the chosen particle itself. Additionally, given that the chosen particle lies initially on the constant-mass surface ($r = constant$) of the mass-model, $M(r)$, then we replace the static scene itself by the collection of particles contained within this constant-mass surface.

To define perspective information for the chosen particle, we note that there is only one distinguished point in the model universe, and that is the origin of the mass-model. Consequently, the most obvious possibility for perspective information is given by the direction-of-view from the chosen particle onto the mass-model origin. Noting how the specification of a constant-mass surface plus the direction to the origin uniquely fixes the position of the chosen particle in the model universe, we conclude that this particle's perspective of the model universe is completely defined by its constant-mass surface plus its direction-of-view onto the mass-model origin.

Finally, we note that, subject to the magnitude of the normal gradient vector, $|\nabla M|$, being a monotonic function of r, total perspective information is precisely carried by the normal gradient vector itself. This follows since the assumed monotonicity of $|\nabla M|$ means that this magnitude is in a 1:1 relation with r and so can be considered to define *which* constant-mass surface is observed; simultaneously, the direction of ∇M is always radial, and so defines the direction-of-view from the chosen particle onto the mass-model origin.

So, to summarize, the perspective of the chosen particle can be considered defined by the normal gradient vector, $\mathbf{n} \equiv \nabla M$, at the particle's position.

5.3 Change in perspective in the model universe

We now consider the change in perspective arising from an infinitesimal change in coordinate position: defining the components of the normal gradient vector (the perspective) as $n_a \equiv \nabla_a M$, $a = 1, 2, 3$, then the *change* in perspective for a coordinate displacement $d\mathbf{r} \equiv (dx^1, dx^2, dx^3)$ is given by

$$dn_a = \nabla_j(\nabla_a M)dx^j \equiv g_{ja}dx^j, \quad g_{ab} \equiv \nabla_a \nabla_b M, \tag{3}$$

for which it is assumed that the geometrical connections required to give this latter expression an unambiguous meaning will be defined in due course. Given that g_{ab} is non-singular, we now note that (3) provides a 1:1 relationship between the contravariant vector dx^a (defining change in the observer's coordinate position) and the covariant vector dn_a (defining the corresponding change in the observer's perspective). It follows that we can define dn_a as the covariant form of dx^a, so that g_{ab} automatically becomes the mass model metric tensor. The scalar product $dS^2 \equiv dn_i dx^i$ is then the absolute magnitude of the coordinate displacement, dx^a, defined relative to the change in perspective arising from the coordinate displacement.

The units of dS^2 are easily seen to be those of *mass* only and so, in order to make them those of $length^2$ - as dimensional consistency requires - we define the working invariant as $ds^2 \equiv (2r_0^2/m_0)dS^2$, where r_0 and m_0 are scaling constants for the distance and mass scales respectively and the numerical factor has been introduced for later convenience.

Finally, if we want

$$ds^2 \equiv \left(\frac{r_0^2}{2m_0}\right) dn_i dx^i \equiv \left(\frac{r_0^2}{2m_0}\right) g_{ij} dx^i dx^j \tag{4}$$

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to behave sensibly in the sense that $ds^2 = 0$ only when $d\mathbf{r} = 0$, then we must replace the condition of non-singularity of g_{ab} by the condition that it is strictly positive (or negative) definite; in the physical context of the present problem, this will be considered to be a self-evident requirement.

5.4 The connection coefficients

We have assumed that the geometrical connection coefficients can be defined in some sensible way. To do this, we simply note that, in order to define conservation laws (ie to do physics) in a Riemannian space, it is necessary to be have a generalized form of Gausses' divergence theorem in the space. This is certainly possible when the connections are defined to be the metrical connections, but it is by no means clear that it is ever possible otherwise. Consequently, the connections are assumed to be metrical and so g_{ab}, given at (3), can be written explicitly as

$$g_{ab} \equiv \nabla_a \nabla_b M \equiv \frac{\partial^2 M}{\partial x^a \partial x^b} - \Gamma_{ab}^k \frac{\partial M}{\partial x^k}, \tag{5}$$

where Γ_{ab}^k are the Christoffel symbols, and given by

$$\Gamma_{ab}^k = \frac{1}{2} g^{kj} \left(\frac{\partial g_{bj}}{\partial x^a} + \frac{\partial g_{ja}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^j} \right).$$

6 The metric tensor given in terms of the mass model

It is shown, in appendix A, how, for an arbitrarily defined mass model, $M(r)$, (5) can be exactly resolved to give an explicit form for g_{ab} in terms of such a general $M(r)$: Defining

$$\mathbf{r} \equiv (x^1, x^2, x^3), \quad \Phi \equiv \frac{1}{2} < \mathbf{r} | \mathbf{r} > \quad \text{and} \quad M' \equiv \frac{dM}{d\Phi}$$

where $< \cdot | \cdot >$ denotes a scalar product, then it is found that

$$g_{ab} = A\delta_{ab} + Bx^i x^j \delta_{ia} \delta_{jb}, \tag{6}$$

where

$$A \equiv \frac{d_0 M + m_1}{\Phi}, \quad B \equiv -\frac{A}{2\Phi} + \frac{d_0 M' M'}{2A\Phi}.$$

for arbitrary constants d_0 and m_1 where, as inspection of the structure of these expressions for A and B shows, d_0 is dimensionless and m_1 has dimensions of mass. Noting that M always occurs in the form $d_0 M + m_1$, it is convenient to write $\mathcal{M} \equiv d_0 M + m_1$, and to write A and B as

$$A \equiv \frac{\mathcal{M}}{\Phi}, \quad B \equiv -\left(\frac{\mathcal{M}}{2\Phi^2} - \frac{\mathcal{M}'\mathcal{M}'}{2d_0\mathcal{M}} \right). \tag{7}$$

7 An invariant calibration of the radial scale

So far, we have assumed an arbitrary calibration for the radial scale; that is, we have assumed only that $r = f(M)$ where f is an arbitrary monotonic increasing function of the mass, M. We seek to find the calibration that incorporates the physical content (that is, the perspective information) of the metric tensor defined at (6).

7.1 The geodesic radial scale

Using (6) and (7) in (4), and applying the identities $x^i dx^j \delta_{ij} \equiv r dr$ and $\Phi \equiv r^2/2$, we find, for an arbitrary displacement $d\mathbf{x}$, the invariant measure:

$$ds^2 = \left(\frac{r_0^2}{2m_0} \right) \left\{ \frac{\mathcal{M}}{\Phi} dx^i dx^j \delta_{ij} - \Phi \left(\frac{\mathcal{M}}{\Phi^2} - \frac{\mathcal{M}'\mathcal{M}'}{d_0\mathcal{M}} \right) dr^2 \right\},$$

which is valid for the arbitrary calibration $r = f(M)$. If the displacement $d\mathbf{x}$ is now constrained to be purely radial, then we find

$$ds^2 = \left(\frac{r_0^2}{2m_0} \right) \left\{ \Phi \left(\frac{\mathcal{M}'\mathcal{M}'}{d_0\mathcal{M}} \right) dr^2 \right\}.$$

Use of $\mathcal{M}' \equiv d\mathcal{M}/d\Phi$ and $\Phi \equiv r^2/2$ reduces this latter relationship to

$$\begin{aligned} ds^2 &= \frac{r_0^2}{d_0 m_0} \left(d\sqrt{\mathcal{M}} \right)^2 \quad \rightarrow \quad ds = \frac{r_0}{\sqrt{d_0 m_0}} d\sqrt{\mathcal{M}} \quad \rightarrow \\ s &= \frac{r_0}{\sqrt{d_0 m_0}} \left(\sqrt{\mathcal{M}} - \sqrt{\mathcal{M}_0} \right), \quad \text{where} \quad \mathcal{M}_0 \equiv \mathcal{M}(s=0) \end{aligned}$$

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which defines the invariant magnitude of an arbitrary *radial* displacement from the origin purely in terms of the mass-model representation $\mathcal{M} \equiv d_0 M + m_1$. By definition, this s is the radial measure which incorporates the physical content of the metric tensor (6), and so the required calibration is obtained simply by making the identity $r \equiv s$.

To summarize, the natural physical calibration for the radial scale is given by

$$r = \frac{r_0}{\sqrt{d_0 m_0}} \left(\sqrt{\mathcal{M}} - \sqrt{\mathcal{M}_0} \right), \tag{8}$$

where $\mathcal{M}_0$ is the value of $\mathcal{M}$ at $r = 0$.

7.2 The Euclidean metric

Using $\mathcal{M} \equiv d_0 M + m_1$ and noting that $M(r = 0) = 0$ necessarily, then $\mathcal{M}_0 = m_1$ and so (8) can be equivalently arranged as

$$\mathcal{M} = \left[\frac{\sqrt{d_0 m_0}}{r_0} r + \sqrt{m_1} \right]^2. \tag{9}$$

Using $\mathcal{M} \equiv d_0 M + m_1$ again, then the mass-distribution function can be expressed in terms of the invariant radial displacement as

$$M = m_0 \left(\frac{r}{r_0} \right)^2 + 2 \sqrt{\frac{m_0 m_1}{d_0}} \left(\frac{r}{r_0} \right) \tag{10}$$

which, for the particular case $m_1 = 0$ becomes $M = m_0 (r/r_0)^2$. Reference to (6) shows that, with this mass distribution and $d_0 = 1$, then $g_{ab} = \delta_{ab}$ so that the metric space becomes Euclidean. Thus, whilst we have yet to show that a globally inertial space can be associated with a non-trivial global matter distribution (since no temporal dimension, and hence no dynamics has been introduced), we have shown that a globally *Euclidean* space can be associated with a non-trivial matter distribution, and that this distribution is necessarily fractal with $D = 2$.

Note also that, on a large enough scale and for *arbitrary* values of m_1, (10)

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shows that radial distance varies as the square-root of mass from the chosen origin - or, equivalently, the mass varies as r^2. Consequently, on sufficiently large scales Euclidean space is irreducibly related to a quasi-fractal, $D = 2$, matter distributions. Since $M/r^2 \approx m_0/r_0^2$ on a large enough scale then, for the remainder of this paper, the notation $g_0 \equiv m_0/r_0^2$ is employed.

8 The temporal dimension

So far, the concept of 'time' has only entered the discussion in the form of the qualitative definition given in §4 - it has not entered in any quantitative way and, until it does, there can be no discussion of dynamical processes.

Since, in its most general definition, time is a parameter which orders change within a system, then a necessary pre-requisite for its quantitative definition in the model universe is a notion of change within that universe, and the only kind of change which can be defined in such a simple place as the model universe is that of internal change arising from the spatial displacement of particles. Furthermore, since the system is populated solely by primitive particles which possess only the property of enumerability (and hence quantification in terms of the *amount* of material present) then, in effect, all change is gravitational change. This fact is incorporated into the cosmology to be derived by constraining all particle displacements to satisfy the Weak Equivalence Principle. We are then led to a Lagrangian description of particle motions in which the Lagrange density is degree zero in its temporal-ordering parameter. From this, it follows that the corresponding Euler-Lagrange equations form an *incomplete* set.

The origin of this problem traces back to the fact that, because the Lagrangian density is degree zero in the temporal ordering parameter, it is then invariant with respect to any transformation of this parameter which preserves the ordering. This implies that, in general, temporal ordering parameters cannot be identified directly with physical time - they merely share one essential char-

acteristic. This situation is identical to that encountered in the Lagrangian formulation of General Relativity; there, the situation is resolved by defining the concept of 'particle proper time'. In the present case, this is not an option because the notion of particle proper time involves the prior definition of a system of observer's clocks - so that some notion of clock-time is factored into the prior assumptions upon which General Relativity is built.

In the present case, it turns out that the isotropies already imposed on the system conspire to provide an automatic resolution of the problem which is consistent with the already assumed interpretation of 'time' as a measure of ordered change in the model universe. To be specific, it turns out that the elapsed time associated with any given particle displacement is proportional, via a scalar field, to the invariant spatial measure attached to that displacement. Thus, physical time is defined directly in terms of the invariant measures of *process* within the model universe.

9 Dynamical constraints in the model universe

Firstly, and as already noted, the model universe is populated exclusively by primitive particles which possess solely the property of enumeration, and hence quantification. Consequently, all motions in the model universe are effectively gravitational, and we model this circumstance by constraining all such motions to satisfy the Weak Equivalence Principle by which we mean that the trajectory of a body is independent of its internal constitution. This constraint can be expressed as:

> **C1** *Particle trajectories are independent of the specific mass values of the particles concerned;*

Secondly, given the isotropy conditions imposed on the model universe from the chosen origin, symmetry arguments lead to the conclusion that the net action of the whole universe of particles acting on any given single particle is such that

any net acceleration of the particle must always appear to be directed through the coordinate origin. Note that this conclusion is *independent* of any notions of retarded or instantaneous action. This constraint can then be stated as:

> **C2** *Any acceleration of any given material particle must necessarily be along the line connecting the particular particle to the coordinate origin.*

10   Gravitational trajectories

Suppose p and q are two arbitrarily chosen point coordinates on the trajectory of the chosen particle, and suppose that (4) is integrated between these points to give the scalar invariant

$$I(p,q) = \int_p^q \left(\frac{1}{\sqrt{2g_0}} \right) \sqrt{dn_i dx^i} \equiv \int_p^q \left(\frac{1}{\sqrt{2g_0}} \right) \sqrt{g_{ij} dx^i dx^j}. \qquad (11)$$

Then, in accordance with the foregoing interpretation, $I(p,q)$ gives a scalar record of how the particle has moved between p and q defined with respect to the particle's continually changing relationship with the mass model, $M(r)$.

Now suppose $I(p,q)$ is minimized with respect to choice of the trajectory connecting p and q, then this minimizing trajectory can be interpreted as a geodesic in the Riemannian space which has g_{ab} as its metric tensor. Given that g_{ab} is defined in terms of the mass model $M(r)$ - the existence of which is independent of any notion of 'inertial mass', then the existence of the metric space, and of geodesic curves within it, is likewise explicitly independent of any concept of inertial-mass. It follows that the identification of the particle trajectory **r** with these geodesics means that particle trajectories are similarly independent of any concept of inertial mass, and can be considered as the modelling step defining that general subclass of trajectories which conform to that characteristic phenomenology of gravitation defined by condition **C1** of §9.

11 The equations of motion

Whilst the mass distribution, represented by $\mathcal{M}$, has been explicitly determined in terms of the geodesic distance at (9), it is convenient to develop the theory in terms of unspecified $\mathcal{M}$.

The geodesic equations in the space with the metric tensor (6) can be obtained, in the usual way, by defining the Lagrangian density

$$\mathcal{L} \equiv \left(\frac{1}{\sqrt{2g_0}} \right) \sqrt{g_{ij} \dot{x}^i \dot{x}^j} = \left(\frac{1}{\sqrt{2g_0}} \right) \left(A <\dot{\mathbf{r}}|\dot{\mathbf{r}}> +B\dot{\Phi}^2 \right)^{1/2}, \qquad (12)$$

where $\dot{x}^i \equiv dx^i/dt$, etc., and writing down the Euler-Lagrange equations

$$2A\ddot{\mathbf{r}} \;\; + \;\; \left(2A'\dot{\Phi} - 2\frac{\dot{\mathcal{L}}}{\mathcal{L}}A \right)\dot{\mathbf{r}} + \left(B'\dot{\Phi}^2 + 2B\ddot{\Phi} - A' <\dot{\mathbf{r}}|\dot{\mathbf{r}}> -2\frac{\dot{\mathcal{L}}}{\mathcal{L}}B\dot{\Phi} \right)\mathbf{r}$$

$$= \;\; 0, \qquad (13)$$

where $\dot{\mathbf{r}} \equiv d\mathbf{r}/dt$ and $A' \equiv dA/d\Phi$, etc. By identifying particle trajectories with geodesic curves, this equation is now interpreted as the equation of motion, referred to the chosen origin, of a single particle satisfying condition **C1** of §9.

However, noting that the variational principle, (11), is of order zero in its temporal ordering parameter, we can conclude that the principle is invariant with respect to arbitrary transformations of this parameter; in turn, this means that the temporal ordering parameter cannot be identified with physical time. This problem manifests itself formally in the statement that the equations of motion (13) do not form a complete set, so that it becomes necessary to specify some extra condition to close the system.

A similar circumstance arises in General Relativity theory when the equations of motion are derived from an action integral which is formally identical to (11). In that case, the system is closed by specifying the arbitrary time parameter to be the 'proper time', so that

$$d\tau = \mathcal{L}(x^j, dx^j) \quad \rightarrow \quad \mathcal{L}(x^j, \frac{dx^j}{d\tau}) = 1, \qquad (14)$$

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which is then considered as the necessary extra condition required to close the system. In the present circumstance, we are rescued by the, as yet, unused condition **C2**.

12 Physical time

12.1 Completion of equations of motion

Consider **C2**, which states that any particle accelerations must necessarily be directed through the coordinate origin. This latter condition simply means that the equations of motion must have the general structure

$$\ddot{\mathbf{r}} = G(t, \mathbf{r}, \dot{\mathbf{r}})\mathbf{r},$$

for scalar function $G(t, \mathbf{r}, \dot{\mathbf{r}})$. In other words, (13) satisfies condition **C2** if the coefficient of $\dot{\mathbf{r}}$ is zero, so that

$$\left(2A'\dot{\Phi} - 2\frac{\dot{\mathcal{L}}}{\mathcal{L}}A\right) = 0 \quad \rightarrow \quad \frac{A'}{A}\dot{\Phi} = \frac{\dot{\mathcal{L}}}{\mathcal{L}} \quad \rightarrow \quad \mathcal{L} = k_0 A, \tag{15}$$

for arbitrary constant k_0 which is necessarily positive since $A > 0$ and $\mathcal{L} > 0$. The condition (15), which guarantees (**C2**), can be considered as the condition required to close the incomplete set (13), and is directly analogous to (14), the condition which defines 'proper time' in General Relativity.

12.2 Physical time defined as process

Equation (15) can be considered as that equation which removes the pre-existing arbitrariness in the 'time' parameter by *defining* physical time:- from (15) and (12) we have

$$\mathcal{L}^2 = k_0^2 A^2 \rightarrow A < \dot{\mathbf{r}}|\dot{\mathbf{r}} > + B\dot{\Phi}^2 = 2g_0 k_0^2 A^2 \rightarrow$$
$$g_{ij}\dot{x}^i \dot{x}^j = 2g_0 k_0^2 A^2 \tag{16}$$

so that, in explicit terms, physical time is *defined* by the relation

$$dt^2 = \left(\frac{1}{2g_0 k_0^2 A^2}\right) g_{ij} dx^i dx^j, \quad \text{where} \quad A \equiv \frac{\mathcal{M}}{\Phi}. \tag{17}$$

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In short, the elapsing of time is given a direct physical interpretation in terms of the process of *displacement* in the model universe.

Finally, noting that, by (17), the dimensions of k_0^2 are those of $L^6/[T^2 \times M^2]$, then the fact that $g_0 \equiv m_0/r_0^2$ (cf §7) suggests the change of notation $k_0^2 \propto v_0^2/g_0^2$, where v_0 is a constant having the dimensions (but not the interpretation) of 'velocity'. So, as a means of making the dimensions which appear in the development more transparent, it is found convenient to use the particular replacement $k_0^2 \equiv v_0^2/(4d_0^2 g_0^2)$, where d_0 is the dimensionless global constant introduced in §6. With this replacement, the *definition* of physical time, given at (17), becomes

$$ dt^2 = \left(\frac{4d_0^2 g_0}{v_0^2 A^2} \right) g_{ij} dx^i dx^j. \tag{18} $$

Since, as is easily seen from the definition of g_{ab} given in §6, $g_{ij} dx^i dx^j$ is necessarily finite and non-zero for a non-trivial displacement $d\mathbf{r}$

12.3 The necessity of $v_0^2 \neq 0$

Equation (18) provides a definition of physical time in terms of basic process (displacement) in the model universe. Since the parameter v_0^2 occurs no where else, except in its explicit position in (18), then it is clear that setting $v_0^2 = 0$ is equivalent to physical time becoming undefined. Therefore, of necessity, $v_0^2 \neq 0$ and all non-zero finite displacements are associated with a non-zero finite elapsed physical time.

13 The cosmological potential

The model is most conveniently interpreted when expressed in potential terms and so, in the following, it is shown how this is done.

13.1 The equations of motion: potential form

From §12, when (15) is used in (13) there results

$$ 2A\ddot{\mathbf{r}} + \left(B'\dot{\Phi}^2 + 2B\ddot{\Phi} - A' <\dot{\mathbf{r}}|\dot{\mathbf{r}}> -2\frac{A'}{A}B\dot{\Phi}^2 \right) \mathbf{r} = 0. \tag{19} $$

Suppose we define a function V according to $V \equiv C_0 - <\dot{\mathbf{r}}|\dot{\mathbf{r}}>/2$, for some arbitrary constant C_0; then, by (16)

$$V \equiv C_0 - \frac{1}{2}<\dot{\mathbf{r}}|\dot{\mathbf{r}}> = C_0 - \frac{v_0^2}{4d_0^2 g_0}A + \frac{B}{2A}\dot{\Phi}^2, \tag{20}$$

where A and B are defined at (7). With unit vector, $\hat{\mathbf{r}}$, then appendix B shows how this function can be used to express (19) in the potential form

$$\ddot{\mathbf{r}} = -\frac{dV}{dr}\hat{\mathbf{r}} \tag{21}$$

so that V is a potential function, and C_0 is the arbitrary constant usually associated with a potential function.

13.2   The potential function, V, as a function of r

From (20), we have

$$2C_0 - 2V = \dot{r}^2 + r^2\dot{\theta}^2 = \frac{v_0^2}{2d_0^2 g_0}A - \frac{B}{A}r^2\dot{r}^2$$

so that V is effectively given in terms of r and $\dot{r}$. In order to clarify things further, we now eliminate the explicit appearance of $\dot{r}$. Since all forces are central, then angular momentum is conserved; consequently, after using conserved angular momentum, h, and the definitions of A, B and $\mathcal{M}$ given in §6, the foregoing equations can be written as

$$
\begin{aligned}
2C_0 \quad - \quad 2V \quad = \quad & \\
\dot{r}^2 \quad + \quad r^2\dot{\theta}^2 = \quad & v_0^2 + \frac{4v_0^2}{r}\sqrt{\frac{m_1}{d_0 g_0}} + \frac{d_0-1}{r^2}\left(\frac{6m_1 v_0^2}{d_0^2 g_0} - h^2\right) \\
+ \quad & \frac{2}{r^3}\sqrt{\frac{d_0 m_1}{g_0}}\left(\frac{2m_1 v_0^2}{d_0^2 g_0} - h^2\right) + \frac{1}{r^4}\frac{m_1}{g_0}\left(\frac{m_1 v_0^2}{d_0^2 g_0} - h^2\right) \tag{22}
\end{aligned}
$$

so that $V(r)$ is effectively given by the right-hand side of (22).

14   A discussion of the potential function

It is clear from (22) that m_1 plays the role of the mass of the central source which generates the potential, V. A relatively detailed description of the behaviour of

V is given in appendix C, where we find that there are two distinct classes of solution depending on the free parameters of the system. These classes can be described as:

- A constant potential universe within which all points are dynamically indistinguishable; this corresponds to an inertial material universe, and arises in the case $m_1 = 0, \ d_0 = 1$;

- All other possibilities give rise to a 'distinguished origin' universe in which either:

  - there is a singularity at the centre, $r = 0$;

  - or there is no singularity at $r = 0$ and, instead, the origin is the centre of a non-trivial sphere of radius $R_{min} > 0$ which acts as an impervious boundary between the exterior universe and the potential source. In effect, this sphere provides the source with a non-trivial spatial extension so that the classical notion of the massive point-source is avoided.

Of these possibilities, the constant potential universe is the one which provides positive answers to our originally posed questions, and it is this which is discussed in detail in the following sections.

However, of the two cases in the distinguished origin universe, the no-singularity case offers the interesting possibility of being able to model the gravitational effects created by a central massive source, but without the non-physical singularity at the origin. This case is mentioned here for future reference.

15   The fractal $D\!=\!2$ inertial universe

Reference to (22) shows that the parameter choice $m_1 = 0$ and $d_0 = 1$ makes the potential function constant everywhere, whilst (10) shows how, for this case, universal matter in an equilibrium universe is necessarily distributed as an exact fractal with $D = 2$. Thus, the fractal $D = 2$ material universe is necessarily a

globally inertial equilibrium universe, and the questions originally posed in §1 are finally answered.

15.1 Implications for theories of gravitation

Given that gravitational phenomena are usually considered to arise as mass-driven perturbations of flat inertial backgrounds, then the foregoing result - to the effect that the inertial background is necessarily associated with a non-trivial fractal matter distribution - must necessarily give rise to completely new perspectives about the nature and properties of gravitational phenomena. However, as we show in §15.2, the kinematics in this inertial universe is unusual, and suggests that the inertial material distribution is more properly interpreted as a quasi-photon fractal gas out of which (presumably) we can consider ordinary material to condense in some fashion.

15.2 The quasi-photon fractal gas

For the case $m_1 = 0$, $d_0 = 1$, the definition M at (10) together with the definitions of A and B in §6 give

$$A = \frac{2m_0}{r_0^2}, \quad B = 0$$

so that, by (20) (remembering that $g_0 \equiv m_0/r_0^2$) we have

$$< \dot{\mathbf{r}} | \dot{\mathbf{r}} > = v_0^2 \tag{23}$$

for all displacements in the model universe. It is (almost) natural to assume that the constant v_0^2 in (23) simply refers to the constant velocity of any given particle, and likewise to assume that this can differ between particles. However, each of these assumptions would be wrong since - as we now show - v_0^2 is, firstly, more properly interpreted as a conversion factor from spatial to temporal units and, secondly, is a *global* constant which applies equally to all particles.

To understand these points, we begin by noting that (23) is a special case

of (16) and so, by (17), is more accurately written as

$$dt^2 = \frac{1}{v_0^2} < d\mathbf{r}|d\mathbf{r} > \tag{24}$$

which, by the considerations of §12.2, we recognize as the *definition* of the elasped time experienced by any particle undergoing a spatial displacement $d\mathbf{r}$ in the model inertial universe. Since this universe is isotropic about all points, then there is nothing which can distinguish between two separated particles (other than their separateness) undergoing displacements of equal magnitudes; consequently, each must be considered to have experienced equal elapsed times. It follows from this that v_0^2 is not to be considered as a locally defined particle velocity, but is a *globally* defined constant which has the effect of converting between spatial and temporal units of measurement.

We now see that the model inertial universe, with (24) as a global relationship, bears a close formal resemblance to a universe filled purely with Einsteinien photons - the difference being, of course, that the particles in the model inertial universe are assumed to be countable and to have mass properties. This formal resemblance means that the model inertial universe can be likened to a quasi-photon fractal gas universe.

16 A quasi-fractal mass distribution law, $M \approx r^2$: the evidence

A basic assumption of the *Standard Model* of modern cosmology is that, on some scale, the universe is homogeneous; however, in early responses to suspicions that the accruing data was more consistent with Charlier's conceptions of an hierarchical universe (Charlier, 1908, 1922, 1924) than with the requirements of the *Standard Model*, de Vaucouleurs (1970) showed that, within wide limits, the available data satisfied a mass distribution law $M \approx r^{1.3}$, whilst Peebles (1980) found $M \approx r^{1.23}$. The situation, from the point of view of the *Standard Model*, has continued to deteriorate with the growth of the data-base to the point that, (Baryshev et al (1995))

> *...the scale of the largest inhomogeneities (*discovered to date*) is comparable with the extent of the surveys, so that the largest known structures are limited by the boundaries of the survey in which they are detected.*

For example, several recent redshift surveys, such as those performed by Huchra et al (1983), Giovanelli and Haynes (1986), De Lapparent et al (1988), Broadhurst et al (1990), Da Costa et al (1994) and Vettolani et al (1994) etc have discovered massive structures such as sheets, filaments, superclusters and voids, and show that large structures are common features of the observable universe; the most significant conclusion to be drawn from all of these surveys is that the scale of the largest inhomogeneities observed is comparable with the spatial extent of the surveys themselves.

In recent years, several quantitative analyses of both pencil-beam and wide-angle surveys of galaxy distributions have been performed: three recent examples are give by Joyce, Montuori & Labini (1999) who analysed the CfA2-South catalogue to find fractal behaviour with $D = 1.9 \pm 0.1$; Labini & Montuori (1998) analysed the APM-Stromlo survey to find fractal behaviour with $D = 2.1 \pm 0.1$, whilst Labini, Montuori & Pietronero (1998) analysed the Perseus-Pisces survey to find fractal behaviour with $D = 2.0 \pm 0.1$. There are many other papers of this nature in the literature all supporting the view that, out to medium depth at least, galaxy distributions appear to be fractal with $D \approx 2$.

This latter view is now widely accepted (for example, see Wu, Lahav & Rees (1999)), and the open question has become whether or not there is a transition to homogeneity on some sufficiently large scale. For example, Scaramella et al (1998) analyse the ESO Slice Project redshift survey, whilst Martinez et al (1998) analyse the Perseus-Pisces, the APM-Stromlo and the 1.2-Jy IRAS redshift surveys, with both groups finding evidence for a cross-over to homogeneity at large scales. In response, the Scaramella et al analysis has been criticized on various grounds by Joyce et al (1999).

So, to date, evidence that galaxy distributions are fractal with $D \approx 2$ on small to medium scales is widely accepted, but there is a lively open debate over the existence, or otherwise, of a cross-over to homogeneity on large scales.

To summarize, there is considerable debate centered around the question of whether or not the material in the universe is distributed fractally or not, with supporters of the big-bang picture arguing that, basically, it is not, whilst the supporters of the fractal picture argue that it is with the weight of evidence supporting $D \approx 2$. This latter position corresponds exactly with the picture predicted by the present approach.

17 Summary and Conclusions

Prompted by the questions

Is it possible to associate a globally inertial space & time with a non-trivial global matter distribution and, if it is, what are the fundamental properties of this distribution?

we have analysed a very simple model universe, consisting solely of an infinite ensemble of particles, possessing only the property of enumerability, existing in a formless continuum and with the ensemble being in a statistically stationary state. No concepts of rods or clocks were imported into this system, and we required that invariant meanings for spatial and temporal intervals should arise from within the ensemble itself.

The notion of the spatial displacement of a particle was given meaning using our common experience - according to which we recognize our own spatial displacements, and their magnitudes, by making reference to our changed views of our local environment and the magnitudes of such changes - and not by referral to formal measuring rods. The formal modelling of this experience led, in §7.2, to the conclusions that, within the model universe:

- On sufficiently large scales, space is necessarily Euclidean (to any required degree of approximation) and is irreducibly associated with a quasi-fractal, $D = 2$, distribution of material within the model universe.

- In the ideal limiting case of a particular parameter going to zero, space is necessarily identically Euclidean and is irreducibly related to a fractal, $D = 2$, distribution of material within the model universe.

This procedure then led, via symmetry arguments, to a formal definition of 'elapsed time' within the model universe as an invariant measure of *ordered process* within that universe. It is to be noted that this is in accord with the way in which we actually experience the passage of time in our lives - as the accumulation of ordered process, and not by continual reference to formal cyclic clocks.

With these definitions of invariant spatial displacement and invariant elapsed time in place, we were then able to answer the original two questions within the context of the model universe so that, finally, we could say:

- On sufficiently large scales, space & time is necessarily inertial (to any required degree of approximation), and is irreducibly associated with a quasi-fractal, $D = 2$, distribution of material within the model universe;

- In the ideal limiting case of a particular parameter going to zero, a globally inertial space & time is irreducibly related to a fractal, $D = 2$, distribution of material within the model universe.

However, the latter ideal inertial universe is distinguished in the sense that whilst all the particles within it have arbitrarily directed motions, the particle velocities all have *equal* magnitude. In this sense, the globally inertial model universe is more accurately to be considered as a quasi-photon gas universe than the universe of our macroscopic experience. In other words, it looks more like a crude model of a material vacuum than the universe of our direct experience.

This result is to be compared with the distribution of galaxies in our directly observable universe which approximates very closely perfectly inertial

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conditions, and which appears to be fractal with $D \approx 2$ on the small-to-medium scale at least. If we make the simple assumption that the distribution of ponderable matter traces the distribution of the material vacuum then, given the extreme simplicity of the analysed model, this latter correspondence between between the model's statements and the cosmic reality lends strong support to the idea that our intuitively experienced perceptions of physical space and time are projected out of relationships, and changing relationships, between the particles (whatever these might be) in the material universe in very much the way described.

The foregoing considerations have fundamental consequences for gravitation theory: specifically, since gravitational phenomena are conventionally considered to arise as mass-driven perturbations of a flat inertial background, then the phenomonology predicted by the analysis - that a flat inertial background is irreducibly associated with a non-trivial fractal distribution of material - must necessarily lead to novel insights into the nature and causes of gravitational phenomena.

Finally, as we have noted, the restriction that the ensemble should be statistically stationary (imposed initially for simplicity) was equivalent to making the analysis non-relativistic. The relativistic counterpart of the foregoing analysis arises from a consideration of a non-stationary universe, and gives rise to a model universe in which the flat spacetime of special relativity is irreducible associated with a relativistically invariant material vacuum of fractal dimension.

A A Resolution of the Metric Tensor

The general system is given by

$$g_{ab} = \frac{\partial^2 M}{\partial x^a \partial x^b} - \Gamma_{ab}^k \frac{\partial M}{\partial x^k},$$

$$\Gamma_{ab}^k \equiv \frac{1}{2} g^{kj} \left(\frac{\partial g_{bj}}{\partial x^a} + \frac{\partial g_{ja}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^j} \right),$$

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and the first major problem is to express g_{ab} in terms of the reference scalar, M. The key to this is to note the relationship

$$\frac{\partial^2 M}{\partial x^a \partial x^b} = M'\delta_{ab} + M''x^a x^b,$$

where $M' \equiv dM/d\Phi$, $M'' \equiv d^2M/d\Phi^2$ and $\Phi \equiv < \mathbf{r}|\mathbf{r} > /2$, since this immediately suggests the general structure

$$g_{ab} = A\delta_{ab} + Bx^a x^b, \tag{25}$$

for unknown functions, A and B. It is easily found that

$$g^{ab} = \frac{1}{A}\left[\delta_{ab} - \left(\frac{B}{A + 2B\Phi}\right)x^a x^b\right]$$

so that, with some effort,

$$\Gamma^k_{ab} = \frac{1}{2A}H_1 - \left(\frac{B}{2A(A + 2B\Phi)}\right)H_2$$

where

$$
\begin{aligned}
H_1 &= A'(x^a \delta_{bk} + x^b \delta_{ak} - x^k \delta_{ab}) \\
&+ B'x^a x^b x^k + 2B\delta_{ab}x^k
\end{aligned}
$$

and

$$
\begin{aligned}
H_2 &= A'(2x^a x^b x^k - 2\Phi x^k \delta_{ab}) \\
&+ 2\Phi B'x^a x^b x^k + 4\Phi Bx^k \delta_{ab}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
g_{ab} &= \frac{\partial^2 M}{\partial x^a \partial x^b} - \Gamma^k_{ab}\frac{\partial M}{\partial x^k} \equiv \delta_{ab}M'\left(\frac{A + A'\Phi}{A + 2B\Phi}\right) \\
&+ x^a x^b \left(M'' - M'\left(\frac{A' + B'\Phi}{A + 2B\Phi}\right)\right).
\end{aligned}
$$

Comparison with (25) now leads directly to

$$
\begin{aligned}
A &= M'\left(\frac{A + A'\Phi}{A + 2B\Phi}\right) = M'\left(\frac{(A\Phi)'}{A + 2B\Phi}\right), \\
B &= M'' - M'\left(\frac{A' + B'\Phi}{A + 2B\Phi}\right).
\end{aligned}
$$

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The first of these can be rearranged as

$$B = \frac{M'}{2\Phi}\left(\frac{(A\Phi)'}{A}\right) - \frac{A}{2\Phi}$$

or as

$$\left(\frac{M'}{A + 2B\Phi}\right) = \frac{A}{(A\Phi)'},$$

and these expressions can be used to eliminate B in the second equation. After some minor rearrangement, the resulting equation is easily integrated to give, finally,

$$A \equiv \frac{d_0 M + m_1}{\Phi}, \quad B \equiv -\frac{A}{2\Phi} + \frac{d_0 M' M'}{2A\Phi}.$$

B   Conservative Form of Equations of Motion

From (20), we have

$$V \equiv -\frac{1}{2} < \dot{\mathbf{r}}|\dot{\mathbf{r}} > = -\frac{k_0^2 A}{2} + \frac{B}{2A}\dot{\Phi}^2, \tag{26}$$

from which we easily find

$$\frac{dV}{dr} \equiv \frac{\partial V}{\partial r} + \frac{\partial V}{\partial \dot{r}}\frac{\ddot{r}}{\dot{r}}$$

$$= \frac{-k_0^2 A'}{2}r + \frac{\dot{\Phi}^2 r}{2A}\left(B' - \frac{A'B}{A}\right) + \frac{B}{A}\left(r\dot{r}^2 + r^2\ddot{r}\right).$$

Since $\dot{r}^2 + r\ddot{r} = \ddot{\Phi}$, then the above expression leads to

$$\frac{dV}{dr}\hat{\mathbf{r}} = \left(\frac{-k_0^2 A'}{2} + \frac{B'}{2A}\dot{\Phi}^2 - \frac{A'B}{2A^2}\dot{\Phi}^2 + \frac{B}{A}\ddot{\Phi}\right)\mathbf{r}.$$

Writing (21) as

$$2A\ddot{\mathbf{r}} + 2A\frac{dV}{dr}\hat{\mathbf{r}} = 0,$$

and using the above expression, we get the equation of motion as

$$2A\ddot{\mathbf{r}} + \left(-k_0^2 AA' + B'\dot{\Phi}^2 - \frac{A'B}{A}\dot{\Phi}^2 + 2B\ddot{\Phi}\right)\mathbf{r} = 0. \tag{27}$$

Finally, from (26), we have

$$k_0^2 A = \frac{B}{A}\dot{\Phi}^2 + < \dot{\mathbf{r}}|\dot{\mathbf{r}} >,$$

which, when substituted into (27), gives (19).

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C Outline analysis of the potential function

It is quite plain from (22) that, for any $m_1 \neq 0$, then the model universe has a preferred centre and that the parameter m_1 (which has dimensions of mass) plays a role in the potential V which is analogous to the source mass in a Newtonian spherical potential - that is, the parameter m_1 can be identified as the mass of the potential source in the model universe. However, setting $m_1 = 0$ is not sufficient to guarantee a constant potential field since any $d_0 \neq 1$ also provides the model universe with a preferred centre. The role of d_0 is most simply discussed in the limiting case of $m_1 = 0$: in this case, the second equation of (22) becomes

$$\dot{r}^2 + r^2 \dot{\theta}^2 \;=\; v_0^2 - (d_0 - 1)\frac{h^2}{r^2}. \tag{28}$$

If $d_0 < 1$ then $|\dot{\mathbf{r}}| \to \infty$ as $r \to 0$ so that a singularity exists. Conversely, remembering that $v_0^2 > 0$ (cf §12.3) then, if $d_0 > 1$, equation (28) restricts real events to the exterior of the sphere defined by $r^2 = (d_0 - 1)h^2/v_0^2$. In this case, the singularity is avoided and the central 'massless particle' is given the physical property of 'finite extension'. In the more realistic case for which $m_1 > 0$, reference to (22) shows that the $r = 0$ singularity is completely avoided whenever $h^2 > m_1 v_0^2/d_0^2 g_0$ since then a 'finite extension' property for the central massive particle always exists. Conversely, a singularity will necessarily exist whenever $h^2 \leq m_1 v_0^2/d_0^2 g_0$.

In other words, the model universe has a preferred centre when either $m_1 > 0$, in which case the source of the potential is a massive central particle having various properties depending on the value of d_0, or when $m_1 = 0$ and $d_1 \neq 0$.

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